CONSISTENT AND COVARIANT ANOMALIES
IN THE OVERLAP FORMULATION OF CHIRAL GAUGE THEORIES

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ABSTRACT

In this letter we show how the covariant anomaly emerges in the overlap scheme. We also prove that the overlap scheme correctly reproduces the anomaly in the flavour currents such as j_μ^5 in vector like theories like QCD.

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The overlap scheme is a framework for a lattice definition of chiral gauge theories [1]. It has been shown that it passes all perturbative tests as long as no gauge fields are involved in the internal loops [2]. In particular, the calculation of the consistent chiral anomalies in the Yang-Mills [3] as well as gravitational backgrounds [4] have been carried out and shown to give the correct continuum limit. One aspect of anomaly calculations, namely, the source and evaluation of the covariant anomaly still needs some clarification. The distinction between covariant and consistent anomalies was described and explained in the pioneering work of Bardeen and Zumino [5]. Here we consider this old question in the context of lattice regularization. Our approach is to set up a functional differential equation for the vacuum amplitude of interest, the 'chiral determinant'. This variational approach to the lattice regularized amplitude is similar to the heat kernel method adopted by Leutwyler [6] and we hope that it will be helpful to make comparisons with Leutwyler's work as we go along. Another question to be considered here concerns the anomalies in global chiral symmetries such as the flavour singlet $U(1)$ axial symmetry of QCD. This must be examined if the overlap is going to be applied to a lattice formulation of vector–like theories such as QCD.

In the overlap scheme one starts from two Hamiltonians, $H_\pm(A)$, which differ from each other in the sign of a mass–like term. To obtain chiral fermions in $D = 2N$ dimensional Euclidean space the Hamiltonians must describe the propagation of the fermions in a $2N + 1$ dimensional Minkowski space-time in the background of the gauge field A. The 4-dimensional Euclidean space will be the plane sitting at the origin of the 5th coordinate which is assumed to be time-like [2]. Let $|A\pm\rangle$ denote the 'Dirac' ground states of our Hamiltonians (negative energy fermion states are filled). They satisfy the eigenvalue equations

$$H_\pm(A)|A\pm\rangle = E_\pm(A)|A\pm\rangle \tag{1}$$

We assume $|A\pm\rangle$ are normalized $\langle A \pm |A\pm\rangle = 1$. We shall give a general argument which is valid on lattice or continuum but the explicit calculational checks will be carried out in the continuum only.

Under an arbitrary variation of A, eqn.(1) yields

$$(\delta E_+ - \delta H_+)|A+\rangle + (E_+ - H_+)\delta|A+\rangle = 0 \tag{2}$$

whose general solution for $\delta|A+\rangle$ is

$$\delta|A+\rangle = G_+(\delta H_+ - \delta E_+)|A+\rangle + i\,\Delta_+(\delta A, A)|A+\rangle \tag{3}$$

where

$$G_+(A) = \left(1 - |A+\rangle\langle A+|\right) \frac{1}{E_+ - H_+} \left(1 - |A+\rangle\langle A+|\right)$$

Since $G_+|A+\rangle = 0 = \langle A+|G_+$ it follows that Δ_+ can be obtained by multiplying (1) by $\langle A+|$,

$$i\,\Delta_+(\delta A, A) = \langle A+|\delta|A+\rangle \tag{4}$$

It is easy to see that $\Delta(\delta A, A)$ is a real quantity. Similar formulae are valid for $|A-\rangle$. We can therefore write the variation of the overlap $\langle A+|A-\rangle$,

$$\delta\,\ell n\langle A+|A-\rangle = L(\delta A, A) + \ell(\delta A, A) \tag{5}$$

where

$$L(\delta A, A) = \frac{\langle A+|\delta H_+ G_+ + G_-\delta H_-|A-\rangle}{\langle A+|A-\rangle} \tag{5.a}$$

$$\ell(\delta A, A) = -i\left(\Delta_+(\delta A, A) - \Delta_-(\delta A, A)\right) \tag{5.b}$$

From these differential forms one can read out the overlap expressions for the currents, L_μ and ℓ_μ which we define to be the coeficients of δA_μ in L and ℓ respectively. It is clear that L, and hence L_μ, does not depend on the choice of the phases for the states $|A\pm\rangle$. (This is not true for ℓ and ℓ_μ.) Our claim is that L_μ reduces, in the continuum limit, to the covariant current. To show this we need to examine the response of various quantities to local gauge transformations. [1]

Under a local gauge transformation $A \to A^\theta$, the Dirac ground states $|A\pm\rangle$ undergo a transformation of the form

$$e^{iF_\theta}|A\pm\rangle = |A^\theta\pm\rangle e^{i\Phi_\pm(\theta, A)} \tag{6}$$

where $\Phi_\pm$ are real angles and $F_\theta = \sum_n \psi^\dagger(n)\,\theta(n)\,\psi(n)$ is an hermitian operator. From (6) we draw several conclusions.

Firstly, define the effective action $\Gamma(A)$ by the overlap formula

$$e^{-\Gamma(A)} = \frac{\langle A+|A-\rangle}{\langle +|-\rangle}.$$

It follows from (6) that this action functional transforms according to

$$\Gamma(A^\theta) - \Gamma(A) = i\left(\Phi_+(\theta, A) - \Phi_-(\theta, A)\right) \tag{7}$$

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[1] Our L and ℓ should be compared with the analogous quantities in [6]

This indicates that the real part of the effective action is gauge invariant but its imaginary part may not be. In fact, we have shown elsewhere that the right hand side of (7) produces exactly the 'consistent' anomaly [3].

Next, assume θ is infinitesimal and expand (6) to first order, obtaining

$$\delta_\theta |A\pm\rangle = i \ F_\theta |A\pm\rangle - i \ \Phi_\pm |A\pm\rangle \tag{8}$$

This implies

$$\langle A \pm |\delta_\theta |A\pm\rangle = i\langle A \pm |F_\theta|A\pm\rangle - i \ \Phi_\pm \tag{9}$$

or, using the definition (4) of Δ, we can rewrite this in the form

$$\Delta_\pm(\delta_\theta A, A) = F_\pm(\theta, A) - \Phi_\pm(\theta, A) \tag{10}$$

where $F_\pm$ is defined by

$$F_\pm(\theta, A) \equiv \langle A \pm |F_\theta|A\pm\rangle \tag{11}$$

From these equations we obtain

$$
\begin{aligned}
\delta_\theta \Gamma(A) &= i\Big(\Phi_+(\theta, A) - \Phi_-(\theta, A)\Big) \\
&= i\Big(F_+(\theta, A) - F_-(\theta, A)\Big) - i\Big(\Delta_+(\delta_\theta A, A) - \Delta_-(\delta_\theta A, A)\Big)
\end{aligned} \tag{12}
$$

We would like to show that $i(F_+ - F_-)$ is indeed the 'covariant' anomaly. The actual proof of this assertion comes down to a direct calculation of $F_+ - F_-$. Therefore, we shall evaluate $F_\pm$ up to first order terms in θ and second order terms in A. This will produce the leading part of the covariant anomaly in $D = 4$ non–Abelian Yang–Mills theory.

First we remark that $F_\pm$, defined by the diagonal matrix elements (11), are evidently covariant,

$$F_\pm(\theta', A') = F_\pm(\theta, A)$$

where

$$
\begin{aligned}
\theta' &= g\theta g^{-1} \\
A' &= gAg^{-1} + gdg^{-1}
\end{aligned}
$$

for any local gauge transformation, $g(n)$. This follows because the transformation is realized by the action of a unitary operator, $exp[iF_\omega]$, on the fermion field operators,

$$e^{-iF_\omega}\psi(n) \ e^{iF_\omega} = g(n)\psi(n)$$

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where F_ω is an appropriate bilinear analogous to F_θ. More generally, it is quite simple to see that $L(\delta A^\theta, A^\theta) = L(\delta A, A)$ and hence that the current L_μ is covariant.

To prove the above assertion we make a perturbative calculation in the continuum. We start from the perturbative expression [3] for $|A+\rangle$

$$|A+\rangle = \alpha_+(A)\left[1 - G_+(V - \Delta E_+)\right]^{-1}|+\rangle \tag{13}$$

where $G_+ = \frac{1-|+\rangle\langle+|}{E_+ - H_+}$ and $V = i \int d^4x\, \psi^\dagger \slashed{A}\, \gamma_5\, \psi$ Here $H_+ = \int d^4x\, \psi^\dagger\, \gamma_5(\slashed{\partial} + \Lambda)\psi$ and $H_+|+\rangle = E_+|+\rangle$. $\alpha_+(A)$ is determined from the normalization condition $\langle A + |A+\rangle = 1$. We shall adopt the Brillouin–Wigner phase convention according to which $\alpha_+(A) > 0$. Since V is first order in A upto the desired order F_+ will be given by

$$
\begin{aligned}
F_+ &= \alpha_+^2\Big[\langle+|F_\theta|+\rangle + 1^{st}\ \text{order} + \\
&\quad \langle+|F_\theta G_+ V G_+ V + V G_+ F_\theta G_+ V + V G_+ V G_+ F_\theta|+\rangle\Big]
\end{aligned} \tag{14}
$$

In writing this expression we have assumed that the first order contributions to ΔE are zero (by Lorentz invariance in the continuum limit).

Define the second order terms $F_+^{(2)}$ by:

$$F_+^{(2)} = \langle+|F_\theta G_+ V G_+ V + V G_+ F_\theta G_+ V + V G_+ V G_+ F_\theta|+\rangle \tag{15}$$

It is easy to show that up to A^2 terms α_+ is given by $\alpha_+^2(A) = 1 - \langle+|V G_+^2 V|+\rangle$. We can then write the A^2 terms in (14) as

$$F_+ = F_+^{(2)} - \langle+|V G_+^2 V|+\rangle\langle+|F_\theta|+\rangle \tag{16}$$

In all of the subsequent calculations we shall assume that the background gauge field is slowly varying in the scale of Λ^{-1}. We shall show that the second term in (16) will not contribute to $F_+ - F_-$. We shall thus concentrate on the evaluation of $F_+^{(2)}$. This can be done by inserting complete sets of states in (15). Here we shall sketch some of the calculational steps. After some relatively simple but lengthy calculations [2] we arrive at

$$
\begin{aligned}
F_+^{(2)} &= \int \left(\frac{dk}{2\pi}\right)^4 \left(\frac{dk'}{2\pi}\right)^4 \left(\frac{dp}{2\pi}\right)^4 \frac{1}{(\omega_k + \omega_p)(\omega_{k'} + \omega_p)} \\
&\quad Tr\Big(\tilde{\theta}(p-k)U(k)\tilde{A}(k-k')U(k')\tilde{A}(k'-p)V(p) \\
&\quad +\tilde{A}(p-k)U(k)\tilde{\theta}(k-k')U(k')\tilde{A}(k'-p)V(p)
\end{aligned}
$$

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[2] The evaluation of the matrix elements have been discussed in more detail in our earlier papers [3].

$$+\tilde{A}(p-k')U(k')\tilde{A}(k'-k)U(k)\tilde{\theta}(k-p)V(p)$$

$$-\tilde{\theta}(k-p)U(p)\tilde{A}(p-k')V(k')\tilde{A}(k'-k)V(k)$$

$$-\tilde{A}(k-p)U(p)\tilde{A}(p-k')V(k')\tilde{\theta}(k'-k)V(k)$$

$$-\tilde{A}(k-k')V(k')\tilde{A}(k'-p)U(p)\tilde{\theta}(p-k)V(k)\Big) \tag{17a}$$

where $U(k) = [\omega_k + \gamma_5(i\slashed{k} + \Lambda)]/2\omega_k = 1 - V(k)$ and $\omega_k = (k^2 + \Lambda^2)^{1/2}$. In evaluating the traces over the γ–matrices we need to consider only the odd powers of Λ. (The even powers will cancel from $F_+ - F_-$.) After further lengthy calculations we arrive at a relatively simple expression for the *pseudoscalar* part of $F_+^{(2)}$,

$$
F_+^{(2)} = -\Lambda \int \left(\frac{dk}{2\pi}\,\frac{dk'}{2\pi}\,\frac{dp}{2\pi}\right)^4 \frac{\varepsilon_{\mu\lambda\nu\sigma}}{\omega_k\omega_{k'}\omega_p}\, tr\Big(\tilde{\theta}(p-k)\tilde{A}_\mu(k-k')\tilde{A}_\nu(k'-p)\Big)
$$

$$
\left[\frac{k_\sigma p_\lambda + k'_\sigma k_\lambda}{(\omega_k + \omega_p)(\omega_{k'} + \omega_p)} + \frac{k_\lambda k'_\sigma + k'_\lambda p_\sigma + k_\sigma p_\lambda}{(\omega_p + \omega_{k'})(\omega_k + \omega_{k'})}\right.
$$
$$
\left. + \frac{k_\lambda k'_\sigma + k_\sigma p_\lambda + k'_\lambda p_\sigma}{(\omega_k + \omega_p)(\omega_{k'} + \omega_k)}\right] + \ldots \tag{17b}
$$

We shall perform the integration over p in the limit in which A_μ and θ are slowly varying over distances of the order of Λ^{-1}. It is more convenient to introduce k_1 and k_2 through $k = p + k_1 + k_2$ and $k' = p + k_2$. We can then write

$$
F_+^{(2)} - F_-^{(2)} = \int \left(\frac{dk_1}{2\pi}\,\frac{dk_2}{2\pi}\right)^4 tr\Big(\tilde{\theta}(-k_1 - k_2)\,\tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2)\Big)\,G_{\mu\nu}(k_1, k_2) \tag{17c}
$$

where in the limit of $\frac{|k_i|}{\Lambda} \ll 1$, $i = 1, 2$; $G_{\mu\nu}(k_1, k_2) = -\frac{\Lambda}{|\Lambda|}\frac{1}{8\pi^2}\,\varepsilon_{\mu\lambda\nu\sigma}k_{1\lambda}k_{2\sigma}$. Thus

$$
F_+^{(2)} - F_-^{(2)} = \frac{1}{8\pi^2}\frac{\Lambda}{|\Lambda|}\int d^4x\,\varepsilon_{\mu\lambda\nu\sigma}\,tr\Big(\theta(x)\partial_\lambda A_\mu \partial_\sigma A_\nu\Big) \tag{17d}
$$

This is the correct leading order term for the covariant anomaly. The second term of (16) actually vanishes for non–Abelian gauge theories, because in this case $\langle +|F_\theta|+\rangle = 0$. This follows from the invariance of $F_+(\theta, A = 0)$ with respect to rigid gauge transformations. For $G = U(1)$ it can be non–zero and divergent, in which case we should use a lattice regularization. However, it will be independent of Λ. Thus if non–zero $\langle +|F_\theta|+\rangle = \langle -|F_\theta|-\rangle$. Therefore in the case $G = U(1)$ the contribution of the second term in (16) to $F_+ - F_-$ will be $\langle +|V G_+^2 V|+\rangle - \langle -|V G_-^2 V|-\rangle$. This quantity, being pseudoscalar and blinear in the gauge field, must vanish in the continuum limit. Hence, the second term

in (16) can be discarded in the Abelian case as well. This has been verified by direct calculation.

Vector like theories such as QCD have no anomalies in the currents that couple to vector potentials. However, the axial vector current j_μ^5 associated with the global chiral symmetry of massless quarks *is* anomalous. If this current is defined covariantly then its divergence is given by

$$\partial_\mu j_\mu^5 = \frac{1}{16\pi^2}\ \varepsilon_{\mu\nu\lambda\sigma}\ tr\ F_{\lambda\mu}\ F_{\sigma\nu}$$

It is easy to understand the derivation of this anomaly from the formalism we have developed above. To this end, assume that the gauge group has the product structure, $G = G_{\text{local}} \times G_{\text{global}}$. In this case the infinitesimal gauge transformation parameter will be the sum of two terms, $\theta = \theta_{\text{local}} + \theta_{\text{global}}$, with θ_{local} and θ_{global} lying in the corresponding Lie algebras. Likewise, the vector potential would be written $A = A_{\text{local}} + A_{\text{global}}$ but with the understanding that A_{global} will be set equal to zero, and the infinitesimal parameter, θ_{global} will be constant (independent of x). Make these substitutions in the formula (17.d), or rather, its fully covariantized generalization,

$$
\begin{aligned}
F_+ - F_- &= \frac{1}{32\pi^2}\ \frac{\Lambda}{|\Lambda|} \int d^4x\ \varepsilon_{\mu\lambda\nu\sigma}\ tr\theta F_{\lambda\mu}F_{\sigma\nu} \\
&= \frac{1}{32\pi^2}\ \frac{\Lambda}{|\Lambda|} \int d^4x\ \varepsilon_{\mu\lambda\nu\sigma}\ tr\Big((\theta_{\text{local}}(x) + \theta_{\text{global}}\ )F_{\text{local}\ \lambda\mu}(x)F_{\text{local}\ \sigma\nu}(x)\Big) (18)
\end{aligned}
$$

The trace in these equations is understood to be taken over the representation of $G_{\text{local}} \times G_{\text{global}}$ to which the fermions belong. The coefficient of θ_{global} defines the global anomaly. Clearly for a non Abelian G_{local} , only the $U(1)$ part of θ_{global} can survive in this formula. Because of the assumed direct product structure, $G_{\text{local}} \times G_{\text{global}}$, the non–Abelian part of G_{global} cannot be anomalous. Of course, one could consider other covariant currents, such as the axial vector colour octet in QCD which, by the same logic, would be anomalous.

The curl of $\Delta_+ - \Delta_-$ is another quantity which is of interst. This quantity has been called C in [6] and is given by

$$C = \pm\ \frac{1}{2\pi} \int d^2x\ \varepsilon_{\mu\nu}\ tr(\delta_2 A_\mu \delta_1 A_\nu) \tag{19a}$$

in 2–dimensions and

$$C = \pm\ \frac{1}{8\pi^2} \int d^4x\ \varepsilon_{\mu\nu\lambda\sigma}\ tr\{\delta_1 A_\mu, \delta_2 A_\nu\}\partial_\lambda A_\sigma \tag{19b}$$

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in 4–dimensions.

Our starting point will be the perturbative expansion (13). To obtain the above results we need to go up to A^3 terms and substitute the result in the definition of $\Delta_+(\delta A, A)$. This series can be used to evaluate the curl of Δ_+. In the first step we get

$$
\begin{aligned}
i\big(\delta_2\Delta_+(\delta_1 A, A) - \delta_1\Delta_+(\delta_2 A, A)\big) = \\
= 2\delta_2\alpha\langle A + |\delta_1(GV) + \delta_1(GV)^2 + \dots|+\rangle \\
+\alpha^2\langle+|\big(\delta_2(VG) + \delta_2(VG)^2 + \dots\big)\big(\delta_1(GV) + \delta_1(GV)^2 + \dots\big)|+\rangle \\
-(1 \leftrightarrow 2) \qquad (20)
\end{aligned}
$$

We have dropped the energy shifts ΔE. If present they will contribute disconnected pieces. Also $\alpha(A)$ contributes disconnected pieces, if it contribute at all to the curl $(\Delta_+ - \Delta_-)$. It is not difficult to see that in $D = 2$ and $D = 4$ the $\delta\alpha$ terms do not contribute to the curl of Δ_+. [3] Equation (20) then reduces to

$$
\begin{aligned}
i \; curl \; \Delta_+ &= i\big(\delta_2\Delta_+(\delta_1 A, A) - \delta_1\Delta_+(\delta_2 A, A)\big) \\
&= \langle+| \big(\delta_2(VG) + \delta_2(VG)^2 + \dots\big) \big(\delta_1(GV) + \delta_1(GV)^2 + \dots\big) |+\rangle \\
&\qquad\qquad -(1 \leftrightarrow 2) \\
&= C_+^{(0)} + C_+^{(1)} + C_+^{(2)} + \dots
\end{aligned}
$$

where

$$
C_+^{(0)} = \langle+|\delta_2(VG)\delta_1(VG)|+\rangle - (1 \leftrightarrow 2) \qquad (21.a)
$$

$$
C_+^{(1)} = \langle+|\delta_2(VG)\delta_1(GV)^2 + \delta_2(VG)^2\delta_1(GV)|+\rangle - (1 \leftrightarrow 2) \qquad (21.b)
$$
$$
\vdots
$$

Our aim is to identify the Λ–odd terms in $C_+^{(i)}$, for slowly varying vector potentials $A_\mu(x)$. The Λ–even terms will be cancelled from $C_+^{(i)} - C_-^{(i)}$. We shall show that $C_+^{(0)}$ will contribute only in $D = 2$ and its contribution will correctly reproduce (19.a). Likewise $C_+^{(1)}$ will contribute in $D = 4$ and its contribution will correctly reproduce (19.b).

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[3]We have shown in [2] that the disconnected contributions take care of themselves and they do not contribute to the effective action.

$\underline{C_+^{(0)}}$: It is not hard to show that

$$C_+^{(0)} = -\int \left(\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\right)^D \frac{1}{(\omega_{k_1}+\omega_{k_2})^2} \left[tr\left(\delta_2\tilde{A}_\mu(k_2-k_1)\delta_1\tilde{A}_\nu(k_1-k_2)\right)\right.$$
$$\left. \cdot tr\left(\gamma_5\gamma_\mu U(k_1)\gamma_5\gamma_\nu V(k_2)\right) - \delta_1\leftrightarrow\delta_2\right]$$

The γ–traces will vanish for $D>2$. For $D=2$ they can easily be evaluated and we obtain

$$C_+^{(0)} = -i\,\varepsilon_{\mu\nu}\,\Lambda\int\left(\frac{dp}{2\pi}\right)^2 tr\,\delta_2\tilde{A}_\mu(-p)G(p)\delta_1\tilde{A}_\nu(p)$$
$$+\Lambda\text{ even terms}$$

where

$$G(p) = \int\left(\frac{dk}{2\pi}\right)^2 \frac{1}{\omega(k+\frac{p}{2})\omega(k-\frac{p}{2})\left(\omega(k+\frac{p}{2})+\omega(k-\frac{p}{2})\right)}$$
$$= \frac{1}{2}\int\left(\frac{dk}{2\pi}\right)^2 \frac{1}{\omega^3(k)} = \frac{1}{4\pi}\cdot\frac{1}{|\Lambda|}$$

where the integral over p has been calculated in the limit of $\frac{|p|}{\Lambda}\to 0$. Thus

$$C_+^{(0)} = -i\,\frac{\Lambda}{|\Lambda|}\frac{1}{4\pi}\int d^2x\,\varepsilon_{\mu\nu}\,tr\,\delta_2 A_\mu(x)\,\delta_1 A_\nu(x)$$

and

Therefore $\delta_2\left(\Delta_+(\delta_1 A, A) - \Delta_-(\delta_1 A, A)\right) - (1\leftrightarrow 2) = C$ where C is given by (19.a).

$\underline{C_+^{(1)}}$: The evaluation of $C_+^{(1)}$ is considerably more lengthy. Firstly it can be shown that as $\Lambda\to\infty$, in $D=2, C_+^{(1)}=0$. Secondly the contribution of the matrix elements of $\delta_2(VG)\delta_1(GV)^2$ in (21.b) in D=4 are given by

$$\langle+|\delta_2(VG)\delta_1(GV)^2|+\rangle =$$
$$= \langle+|\left(\delta_2 VG^2\delta_1 VGV + \delta_2 VG^2 VG\delta_1 V\right)|+\rangle$$

$$\langle+|\delta_2 VG^2\delta_1 VGV|+\rangle - (1\leftrightarrow 2) =$$
$$= -\frac{\Lambda}{|\Lambda|}\frac{i}{96\pi^2}\,\varepsilon_{\mu\nu\lambda\sigma}\int d^4x\,tr\{\delta_2 A_\mu,\delta_1 A_\nu\}\partial_\sigma A_\lambda$$

$$\langle+|\delta_2 VG^2 VG\delta_1 V|+\rangle - (1\leftrightarrow 2) =$$
$$= \frac{\Lambda}{|\Lambda|}\frac{2i}{96\pi^2}\,\varepsilon_{\mu\nu\lambda\sigma}\int d^4x\,tr\{(\delta_1 A_\lambda\delta_2 A_\mu + \delta_2 A_\mu\delta_1 A_\lambda)\partial_\sigma A_\lambda\}$$

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The contribution of the matrix element of $\delta_2(VG)^2\delta_1(GV)$ in (21.b) can be obtained from that of $\delta_2(VG)\delta_1(GV)^2$ by complex conjugation followed by an interchange of $1 \leftrightarrow 2$. We can then assemble all these results to obtain

$$C_+^{(1)} = \frac{\Lambda}{|\Lambda|} \frac{i}{96\pi} \varepsilon_{\mu\nu\lambda\sigma} \int d^4x \; tr\Big[(-2-4)\{\delta_2 A_\mu, \delta_1 A_\nu\}\partial_\sigma A_\lambda\Big]$$

Therefore

$$C_+^{(1)} - C_-^{(1)} = -\frac{\Lambda}{|\Lambda|} \frac{i}{8\pi} \varepsilon_{\mu\nu\lambda\sigma} \int d^4x \; tr\{\delta_2 A_\mu, \delta_1 A_\nu\}\partial_\sigma A_\lambda$$

This is in agreement with the expression (19.b) given by Leutwyler [6].

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